Exercise 1: The neo-Hookean model predicts that the internal pressure p_i in a balloon is given by,

$$p_i(\lambda) = 4C_{10}\frac{e_i}{R}\frac{1}{\lambda}\left(1-\frac{1}{\lambda^6}\right)$$

Here C_{10} is a constant, $\lambda = r/R$ is the stretch ratio and e_i, R are the initial thickness and radius of the balloon. Find the stretch ratio corresponding to maximum pressure in the balloon.

Solution:

The pressure in the inflated balloon is given by equation (**B**6.102)

$$p_i(\lambda) = 4C_{10}\frac{e_i}{R}\frac{1}{\lambda}\left(1-\frac{1}{\lambda^6}\right) .$$

The maximum pressure is obtained when the derivative of p with respect to the stretch ratio λ vanishes

$$\frac{\partial p_i(\lambda)}{\partial \lambda} = 0 .$$

One has

$$\frac{d}{d\lambda}\left(\frac{1}{\lambda}-\frac{1}{\lambda^7}\right) = -\frac{1}{\lambda^2}+\frac{7}{\lambda^8} = 0$$

and thus

$$\lambda^6 = 7 \Rightarrow \lambda = \sqrt[6]{7} = 1.383$$

$$p_i^{max} = 4C_{10}\frac{e_i}{R}\frac{1}{1.383}\left(1-\frac{1}{7}\right) = 2.479\frac{C_{10}e_i}{R} .$$

Exercise 2: A general constitutive relation in non-linear elasticity (for incompressible materials) proposed by Ogden, is,

$$\phi(\lambda_1,\lambda_2,\lambda_3) = \sum_{i=1}^{N}\frac{\mu_i}{\alpha_i}\left(\lambda_1^{\alpha_i}+\lambda_2^{\alpha_i}+\lambda_3^{\alpha_i}-3\right) \qquad \text{(a)}$$

where α_i, μ_i are parameters of the model and N is the number of terms considered in the function. It is chosen according to the experimental data. Show that this model,

(1) reduces to Neo-Hookean model when, $N=1,\ \ \alpha_1=2,\ \ \mu_1/2=C_{10}$ and
(2) reduces Mooney-Rivlin model when
$N=2,\ \ \alpha_1=2,\ \ \alpha_2=-2,\ \ \mu_1/2=C_{10},\ \ -\mu_2/2=C_{01}$.

Solution:

(1) For $N=1$ (a) reduces to ,

$$\Phi = \frac{\mu_1}{\alpha_1}\left(\lambda_1^{\alpha_1} + \lambda_2^{\alpha_1} + \lambda_3^{\alpha_1} - 3\right) \ .$$

Introducing the given values of the parameters and replacing $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = I_1$, we obtain,

$$\Phi = C_{10}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) = C_{10}\left(I_1 - 3\right).$$

(2) For N = 2, the function (a) takes the form

$$\Phi = \frac{\mu_1}{\alpha_1}\left(\lambda_1^{\alpha_1} + \lambda_2^{\alpha_1} + \lambda_3^{\alpha_1} - 3\right) + \frac{\mu_2}{\alpha_2}\left(\lambda_1^{\alpha_2} + \lambda_2^{\alpha_2} + \lambda_3^{\alpha_2} - 3\right).$$

Introducing the given values for the parameters, we obtain,

$$\Phi = C_{10}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + C_{01}\left(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\right)$$

$$\Phi = C_{10}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + C_{01}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3\right) \ .$$

$$= C_{10}\left(I_1 - 3\right) + C_{01}\frac{1}{\lambda_1^2\lambda_2^2\lambda_3^2}\left(\lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2\lambda_2^2 - 3\right)$$

Taking into account the relations of the invariants and stretch ratios

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

and considering incompressibility $I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 = 1$ we have,

$$\Phi = C_{10}\left(I_1 - 3\right) + C_{01}\left(I_2 - 3\right).$$